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Hopf Algebras in Analysis[1,2]

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We present a construction of the Hopf algebra $\mathscr{C}[z]^{\circ}$, dual of the polynomials in one variable, that uses rational functions. This construction illustrates how basic concepts of the theory of bialgebras can be used in analysis. We describe several spaces of interest in analysis that are isomorphic to $\mathscr{C}[z]^{\circ}$. Some of the results presented here were motivated by problems posed by Rota in 1994.

1. INTRODUCTION

Duality in the linear algebra sense is certainly one of the most useful concepts in mathematics. Spaces of linear functionals and pairs of dual vector spaces have been extensively studied, especially the topological aspects and the adjoints of operators. Most of the spaces that appear in analysis are spaces of functions that, in addition to the linear space structure, have other algebraic structure determined, for example, by a multiplication operation or by the action of some group on their elements. Using duality, such additional structure can be transferred to the dual spaces. For example, a multiplication on a space E induces a comultiplication on the dual space E^* and a comultiplication on E induces a multiplication on E^*. This way of applying duality, which is quite natural in algebra, has been rarely used in analysis, probably because translations, difference quotients, and Leibniz-type formulas have not been studied as comultiplications. Buck (1952) used duality to define a multiplication of linear functionals on spaces of continuous functions and, more recently, Brezinski and Maroni (1996) used the algebraic structure of the linear functionals on the polynomials in the theory of Padé approximation.

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In the present paper, we use duality to construct some simple examples of linear spaces that appear often in analysis and that have a bialgebra structure, that is, a multiplication and a comultiplication related in an appropriate way. In fact, most of our examples have a Hopf algebra structure. We use only some basic definitions and simple results about Hopf algebras. On this subject the main references are Abe (1980), Kassel (1995), and Sweedler (1969).

The ideas presented here are an outgrowth of research motivated by questions posed by Rota in 1994. He asked, "Why is it that the convolution of two exponential functions is a difference quotient?" and "Can we find a simple and elegant description of the Hopf algebra dual of the polynomials in one variable?" The second question was proposed by Rota as a homework assignment in a graduate course on Hopf algebras in combinatorics (Rota, 1994). The author was attending the course and conjectured at that time (and proved shortly after) that the Hopf algebra dual of $\mathscr{C}[t]$, usually denoted by $\mathscr{C}[t]^\circ$, is isomorphic to the proper rational functions. Rota considered such a description as an acceptable answer to his question, especially because the description of the comultiplication was very simple. Later we discovered that the abstract Hopf algebra $\mathscr{C}[t]^\circ$ has many concrete realizations that appear naturally in analysis. These are some of the examples that we present in this paper. One of them, the algebra of linearly recurrent sequences, has been studied by several authors (Cerlienco *el al.*, 1987; Chin and Goldman, 1993; Peterson and Taft, 1980). We essentially answered Rota's questions in Verde-Star (1995, 1997) and found simple algebraic explanations of some basic results about convolutions and transform methods for the solution of linear functional equations. See also Verde-Star (2000), where we present a unified method for the solution of linear functional equations.

Rota introduced bialgebras and Hopf algebras in combinatorics (Joni and Rota, 1979). The basic idea is that comultiplications are closely related to decompositions of objects. Hopf algebras are a natural tool for the study of partially ordered sets and other combinatorial objects.

Through his profound questions and his search for simple explanations of the basic facts, Rota motivated the introduction of Hopf algebra ideas in analysis. We believe that Hopf algebras and some natural generalizations will have an important place in analysis.

2. RATIONAL FUNCTIONS

Let f be an entire function. Consider Cauchy's representation formula

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)\,dz}{(z-a)^{k+1}} = \frac{f^{(k)}(a)}{k!} \tag{2.1}$$

where k is in $\mathbb{N}$ and γ is a simple closed contour around a with the usual positive orientation. Formula (2.1) can be interpreted as an integral representation for the linear functional $T_{a,k}$ that sends f to $f^{(k)}(a)/k!$, which we call a *Taylor functional at a of order k*. The kernel function in (2.1) is the rational function

$$r_{a,k}(z) = \frac{1}{(z-a)^{1+k}}, \qquad a \in \mathbb{C}, \quad k \in \mathbb{N} \tag{2.2}$$

Let $\mathcal{R}$ be the complex vector space generated by all the $r_{a,k}$. It is easy to see that $\mathcal{R}$ is the set of all rational functions of the form $p(z)/u(z)$, where p and u are polynomials, and u is monic and has degree strictly greater than the degree of p. The elements of $\mathcal{R}$ are called *proper rational functions*. Let $\mathcal{F}$ denote the linear space of all the entire functions. If we consider (2.1) as an indefinite inner product of $r_{a,k}$ and f, then we can extend it to a map from $\mathcal{R} \times \mathcal{F}$ to $\mathbb{C}$ as follows:

$$\langle h, f \rangle = \frac{1}{2\pi i} \int_\gamma h(z) f(z)\, dz, \qquad h \in \mathcal{R}, \quad f \in \mathcal{F} \tag{2.3}$$

where γ is a positively oriented, simple closed contour whose interior contains all the singularities of h. By the residue theorem, (2.3) can be written as

$$\langle h, f \rangle = \sum_{\alpha \in S(h)} \text{Residue of } h(z) f(z) \text{ at } a \tag{2.4}$$

where $S(h)$ is the set of poles of h. It is clear that the indefinite inner product $\langle\ ,\ \rangle$ can be extended to a larger domain, for example, to $\mathcal{M} \times \mathcal{M}$, where $\mathcal{M}$ denotes the space of meromorphic functions of the form g/u, where $g \in \mathcal{F}$ and u is a polynomial. L. Fantappiè developed from 1923 to 1930 a theory of functionals representable by contour integrals like (2.3), which he called *analytic functionals* (Fantappiè, 1925–26). In what follows, we consider only the indefinite inner product restricted to the rational functions.

For simplicity, we denote by $\mathcal{P}$ the complex vector space of all polynomials in one complex variable and, for any nonnegative integer n, we denote by $\mathcal{P}_n$ the subspace of $\mathcal{P}$ of the polynomials whose degree is at most equal to n. Let $\mathcal{Q}$ be the space of all the rational functions. By the division algorithm for polynomials, $\mathcal{Q}$ is the direct sum of P and R, and thus the union of $\{z^n: n \in \mathbb{N}\}$ and $\{r_{a,k}: a \in \mathbb{C}, k \in \mathbb{N}\}$ is a basis for Q. Allowing h and f to be any elements of $\mathcal{Q}$ in (2.3), we get an antisymmetric bilinear map from $\mathcal{Q} \times \mathcal{Q}$ to $\mathbb{C}$. On the basic elements of $\mathcal{Q}$, we have

$$\langle r_{a,k}(z), z^n \rangle = \binom{n}{k} a^{n-k}, \qquad a \in \mathbb{C}, \quad k, n \in \mathbb{N} \tag{2.5}$$

$$\langle r_{a,k}(z), r_{b,m}(z) \rangle = (-1)^k \binom{k+m}{k} \frac{1}{(a-b)^{1+k+m}}, \qquad a \neq b, \quad k, n \in \mathbb{N} \tag{2.6}$$

$$\langle r_{a,k}, r_{a,m}\rangle = 0, \qquad a \in \mathbb{C}, \quad k, n \in \mathbb{N} \tag{2.7}$$

$$\langle z^n, z^k\rangle = 0, \qquad k, n \in \mathbb{N} \tag{2.8}$$

$$\langle f, g\rangle = -\langle g, f\rangle, \qquad f, g \in \mathcal{Q} \tag{2.9}$$

Let p/u be an element of $\mathcal{R}$ and let g be in $\mathcal{Q}$. Then

$$\left\langle \frac{p}{u}, g \right\rangle = \sum_a \text{Residue of } \left(\frac{pg}{u}\right) \text{at } a \tag{2.10}$$

where the sum runs over the set of the distinct roots of $u(z)$.

Let f and g be rational functions. Then

$$\langle f, pg\rangle = \langle pf, g\rangle, \qquad p \in \mathcal{P} \tag{2.11}$$

$$\langle f, Dg\rangle = \langle -Df, g\rangle \tag{2.12}$$

where D denotes differentiation with respect to z; we have

$$\langle r_{a,n}, fg\rangle = \sum_{k=0}^{n} \langle r_{a,k}, f\rangle \langle r_{a,n-k}, g\rangle \tag{2.13}$$

$$\langle f(z), g(z)\rangle = \langle f(z+a), g(z+a)\rangle, \qquad a \in \mathbb{C} \tag{2.14}$$

Equation (2.13) is Leibniz's rule and (2.14) is the translation invariance property.

We define the *reversion* map R from $\mathcal{Q}$ to $\mathcal{Q}$ by

$$Rf(z) = \frac{1}{z} f\left(\frac{1}{z}\right), \qquad f \in \mathcal{Q} \tag{2.15}$$

Note that $R^2 = I, \langle f, g\rangle = \langle Rg, Rf\rangle$ and $\langle f, Rg\rangle = \langle -Rf, g\rangle$ for any rational functions f and g.

Let p/u be a proper rational function and let q and v be polynomials. Then

$$\left\langle \frac{p}{u}, qv \right\rangle = \left\langle \frac{pq}{u}, v \right\rangle = \left\langle \frac{r}{u}, v \right\rangle \tag{2.16}$$

where $pq = wu + r$ and the degree of r is strictly smaller than the degree of u.

We say that a function f is defined on the roots of a polynomial $u(z)$ if for each root a of u with multiplicity m, the derivatives $D^k f$, for $0 \leq k < m$, are defined at a. Using (2.10), it is easy to prove the following propositions. Let u and v be polynomials and let f be a function that is defined on the roots of uv. Then

$$\left\langle \frac{1}{uv}, vf \right\rangle = \left\langle \frac{1}{u}, f \right\rangle \tag{2.17}$$

and, if u and v have no common roots, then

$$\left\langle \frac{1}{uv}, f \right\rangle = \left\langle \frac{1}{u}, \frac{f}{v} \right\rangle + \left\langle \frac{1}{v}, \frac{f}{u} \right\rangle \tag{2.18}$$

A simple computation yields $u(z) = (z - a)^{1+k}$ and $v(z) = (z - b)^{1+m}$, where $a \neq b$. Then, by (2.18) and (2.13), for any function f defined on the roots of uv, we have

$$\langle r_{a,k}r_{b,m}, f\rangle = \sum_{j=0}^{k} \langle r_{a,j}, r_{b,m}\rangle\langle r_{a,k-j}, f\rangle + \sum_{j=0}^{m} \langle r_{b,j}, r_{a,k}\rangle\langle r_{b,m-j}, f\rangle \tag{2.19}$$

Taking $f(z) = 1/(t - z)$, where $t \neq a$ and $t \neq b$, we get the multiplication formula

$$r_{a,k}r_{b,m} = \sum_{j=0}^{k} \langle r_{a,j}, r_{b,m}\rangle r_{a,k-j} + \sum_{j=0}^{m} \langle r_{b,j}, r_{a,k}\rangle r_{b,m-j} \tag{2.20}$$

This is the basic partial fractions decomposition formula.

Now let $u_j(z) = (z - a_j)^{m_j}$, for $0 \leq j \leq s$, where the a_j are distinct complex numbers and the m_j are positive integers. Let $w = \Pi\, u_j$ and define $q_j = w/u_j$. Let f be a function defined on the roots of w. Then, using the decomposition formula (2.18) repeatedly, we obtain

$$\left\langle \frac{1}{w}, f \right\rangle = \sum_{j=0}^{s} \left\langle \frac{1}{u_j}, \frac{f}{q_j} \right\rangle$$

and by Leibniz' rule we get

$$\left\langle \frac{1}{w}, f \right\rangle = \sum_{j=0}^{s} \sum_{k=0}^{m_j-1} \left\langle r_{a_j,k}, \frac{1}{q_j} \right\rangle \langle r_{a_j,m_j-1-k}, f\rangle \tag{2.21}$$

Taking $f(z) = p(z)/(t - z)$, where p is a polynomial whose degree is less than the degree of w and t is such that $w(t) \neq 0$, we obtain the general *partial fractions decomposition formula*

$$\frac{p(t)}{w(t)} = \sum_{j=0}^{s} \sum_{k=0}^{m_j-1} \left\langle r_{a_j,k}, \frac{p}{q_j} \right\rangle \frac{1}{(t - a_j)^{m_j-k}} \tag{2.22}$$

The linear functional that sends f to $\langle 1/w, f\rangle$ is called the *divided difference* of f with respect to the roots of w. Note that (2.21) gives an explicit expression for this functional as a linear combination of Taylor functionals at the roots of w. If all the multiplicities m_j in (2.21) are equal to one, we get

$$\left\langle \frac{1}{w(z)}\ f(z) \right\rangle = \sum_{k=0}^{s} \frac{f(a_k)}{w'(a_k)}$$

Let $n + 1$ be the degree of w and define $w_{j,k} \in \mathcal{P}_n$,

$$w_{j,k}(t) \ = (t - a_j)^k q_j(t), \qquad 0 \le j \le s, \quad 0 \le k \le m_j - 1$$

$$\left\langle r_{a_j,k}, \frac{w_{i,l}}{q_j} \right\rangle = \delta_{(j,k),(i,l)}$$

This relation implies that the $w_{j,k}$ are linearly independent and hence they are a basis for $\mathcal{P}_n$. The linear functionals that send p to $\langle r_{a_j,k}, p/q_j \rangle$ form the dual basis. By Leibniz' rule, they are linear combinations of the Taylor functionals that correspond to the rational functions $r_{a_j,k}$.

Dividing (2.22) by $w(t)$, we obtain the interpolation formula

$$p(t) = \sum_{j=0}^{s} \sum_{k=0}^{m_j-1} \left\langle r_{a_j,k}, \frac{p}{q_j} \right\rangle w_{j,k}(t), \qquad p \in \mathcal{P}_n$$

3. COALGEBRAS AND BIALGEBRAS

There are many constructions that transform a function of one variable into a function of two variables. For example, if $f(t)$ is a function of the complex variable t, then $f(t + z)$, $f(tz)$, and $(f(t) - f(z))/(t - z)$ are functions of two variables. In some cases, such functions have a weak separation-of-variables property and can be written as a finite sum of products of the form $g_i(t)h_i(z)$. Constructions for which this property holds can be studied from an algebraic point of view and are examples of comultiplications, which we define next. Let k be a field, let $\mathcal{C}$ be a vector space over k, and let i denote the identity map on $\mathcal{C}$. A linear map $\Delta:\mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ is a *comultiplication* if it satisfies the *coassociativity* property

$$(i \otimes \Delta) \circ \Delta = (\Delta \otimes i) \circ \Delta$$

A linear map ϵ: $\mathcal{C} \to k$ is a *counit* if $(\epsilon \otimes i) \circ \Delta$ is the natural isomorphism from $\mathcal{C}$ to $k \otimes \mathcal{C}$ and $(i \otimes \epsilon) \circ \Delta$ is the natural isomorphism from $\mathcal{C}$ to $\mathcal{C} \otimes k$. A space $\mathcal{C}$ with a coassociative comultiplication and a counit is called a *coalgebra*.

In the space $\mathcal{P}$ of polynomials, the map that sends $p(z)$ to $p(t + z)$ may be seen as a comultiplication as follows. Since

$$(t + z)^n = \sum_{j=0}^{n} \binom{n}{j} t^j z^{n-j}$$

we define Δ on the basic elements by

$$\Delta z^n = \sum_{j=0}^{n} \binom{n}{j} z^j \otimes z^{n-j}, \qquad n \geq 0$$

Δ is a comultiplication on $\mathcal{P}$. The counit is $\epsilon p = p(0)$.

Let $\mathcal{C}^*$ be the k-linear dual space of the coalgebra $\mathcal{C}$. The comultiplication Δ induces a multiplication in $\mathcal{C}^*$ as follows:

$$\langle \alpha\beta, f \rangle = \langle \alpha \otimes \beta, \Delta f \rangle, \qquad \alpha, \beta \in \mathcal{C}^*, \quad f \in \mathcal{C}$$

The transpose of ϵ is a unit on $\mathcal{C}^*$. Therefore $\mathcal{C}^*$ is an associative algebra with unit element. In an analogous way, if we start with a finite-dimensional algebra $\mathcal{A}$, then, using duality, we can give $\mathcal{A}^*$ a coalgebra structure. This is not always true if $\mathcal{A}$ is infinite dimensional (Abe, 1980; Kassel, 1995).

A space $\mathcal{H}$ that has both an algebra and a coalgebra structure is called a *bialgebra* if Δ and ϵ are algebra morphisms. This is, if μ and η are the multiplication and the unit of $\mathcal{H}$ and i is the identity map on the field, then $\Delta(fg) = \Delta f \Delta g$, $\Delta \circ \eta = (\eta \otimes \eta) \circ i$, $i \circ (\epsilon \otimes \epsilon) = \epsilon \circ \mu$, and $\epsilon \circ \eta = i$. These properties also mean that μ and η are morphisms of coalgebras. A bialgebra that has an endomorphism S compatible in a certain sense with the bialgebra structure is a Hopf algebra (Kassel, 1995). S is called the antipode map.

An example of Hopf algebra is $\mathcal{P}$. The multiplication is

$$z^j z^k = z^{j+k}, \qquad j, k \in \mathbb{N}$$

and the unit element is $z^0 = 1$. The comultiplication and the counit were defined previously. The antipode is

$$S z^k = (-1)^k z^k, \qquad k \in \mathbb{N}$$

Using the Chu–Vandermonde convolution formula

$$\binom{n + k}{m} = \sum_{j \geq 0} \binom{n}{j} \binom{k}{m - j}$$

it is easy to show that Δ is an algebra map.

If $\mathcal{H}$ is a Hopf algebra, we define

$$\mathcal{H}^\circ = \{\alpha \in \mathcal{H}^* \big| \mu^*(\alpha) \in \mathcal{H}^* \otimes \mathcal{H}^*\}$$

where μ is the multiplication in $\mathcal{H}$ and $\mathcal{H}^*$ is the full linear dual of $\mathcal{H}$. It is easy to see that $\mathcal{H}^\circ$ is a subalgebra of $\mathcal{H}^*$, $\mu^*(\mathcal{H}^\circ) \subset \mathcal{H}^\circ \otimes \mathcal{H}^\circ$, $\Delta^*(\mathcal{H}^\circ \otimes$

$\mathcal{H}^\circ) \subset \mathcal{H}^\circ$, and $S^*(\mathcal{H}^\circ) \subset \mathcal{H}^\circ$. Therefore $\mathcal{H}^\circ$ is a Hopf algebra and it is called the *dual Hopf algebra* of $\mathcal{H}$ or the continuous dual of $\mathcal{H}$. The following proposition is a useful characterization of the dual Hopf algebra.

Proposition 3.1 (Abe, 1980, Theorem 2.2.12). $\mathcal{H}^\circ$ is the set of elements of $\mathcal{H}^*$ whose kernel contains an ideal of $\mathcal{H}$ of finite codimension.

4. THE HOPF DUAL OF THE POLYNOMIALS

The indefinite inner product introduced in Section 2 gives us a large set of linear functionals on the space of polynomials. In particular, each proper rational function f corresponds to the linear functional on $\mathcal{P}$ that sends p to $\langle f, p\rangle$. Therefore we can identify $\mathcal{R}$ with a subspace of $\mathcal{P}^*$. We will show next that the Hopf algebra structure of $\mathcal{P}$ induces by duality a Hopf algebra structure on $\mathcal{R}$ and it is (isomorphic) to $\mathcal{P}^\circ$.

Leibniz' rule (2.13) gives us the comultiplication on $\mathcal{R}$, which we denote by Γ,

$$\Gamma r_{a,n} = \sum_{k=0}^{n} r_{a,k} \otimes r_{a,n-k}, \qquad a \in \mathbb{C}, \quad n \in \mathbb{N}$$

The counit Φ is given by

$$\Phi r_{a,n} = \langle r_{a,n}, 1\rangle = \delta_{0,n}, \qquad a \in \mathbb{C}, \quad n \in \mathbb{N}.$$

The multiplication on $\mathcal{R}$, denoted by $\star$, is determined as follows:

$$\begin{aligned}\langle r_{a,k} \star r_{b,m}, z^n\rangle &= \sum_{j=0}^{n} \binom{n}{j}\langle r_{a,k}, z^j\rangle\langle r_{b,m}, z^{n-j}\rangle \\ &= \sum_{j=0}^{n} \binom{n}{j}\binom{j}{k}a^{j-k}\binom{n-j}{m}b^{n-m-j} \\ &= \binom{k+m}{k}\binom{n}{k+m}(a+b)^{n-k-m}\end{aligned}$$

Therefore

$$r_{a,k} \star r_{b,m} = \binom{k+m}{k} r_{a+b,k+m}$$

This is called the *Hurwitz convolution*. Note that $r_{0,0}$ is the unit element for the multiplication $\star$. For the antipode, we have

$$\langle r_{a,k}(z), Sz^n\rangle = \langle r_{a,k}(z), (-1)^n z^n\rangle = (-1)^n \binom{n}{k} a^{n-k}$$

$$S^* r_{a,k} = (-1)^k r_{-a,k}$$

These definitions give $\mathcal{R}$ the structure of a Hopf algebra.

Theorem 4.1. $\mathcal{R}$ is the set of elements of $\mathcal{P}^*$ whose kernel contains an ideal of $\mathcal{P}$ of finite codimension.

Proof. All the ideals of $\mathcal{P}$ are of the form $w\mathcal{P}$, where w is a polynomial. By the division algorithm for any nonzero w, the quotient $\mathcal{P}/w\mathcal{P}$ has dimension equal to the degree of w. Thus, any nonzero ideal of $\mathcal{P}$ has finite codimension.

Let p/w be an element of $\mathcal{R}$. By (2.17) and (2.8), we have

$$\left\langle \frac{p}{w}, wv \right\rangle = \langle p, v\rangle = 0, \qquad v \in \mathcal{P}$$

and therefore the ideal $w\mathcal{P}$ is contained in the kernel of p/w.

Let α be an element of $\mathcal{P}^*$ such. that the kernel of α contains the ideal $w\mathcal{P}$, where w is a polynomial of degree $n + 1$. Let p be a polynomial. By the division algorithm, $p = qw + u$, where u is a polynomial in $\mathcal{P}_n$. Then $\alpha p = \alpha u$. Therefore, since $\mathcal{P} = w\mathcal{P} \oplus \mathcal{P}_n$, the functional α is determined by its restriction to $\mathcal{P}_n$, call it L. We saw in Section 2 that a basis for the dual space of $\mathcal{P}_n$ consists of the functionals $v \to \langle r_{a_j,k}, v/q_j\rangle$, which are linear combinations of Taylor functionals associated with the roots of w. Therefore L is a linear combination of such functionals and thus corresponds to a proper rational function of the form p/w. ■

If we look again at the definition of the Hopf algebra structure of $\mathcal{R}$, we can see that everything is defined in terms of the indices a and k of the basic elements $r_{a,k}$. This suggests the following definition.

Let $\mathcal{B}$ be the free complex vector space generated by the set $\mathbb{C} \times \mathbb{N}$. Define the multiplication $\star$, the comultiplication Γ, the counit Φ, and the antipode by

$$(a, k) \star (b, m) = \binom{k+m}{k}(a + b, k + m)$$

$$\Gamma(a, n) = \sum_{k=0}^{n} (a, k) \otimes (a, n - k)$$

$$\Phi(a, n) = \delta_{0,n}, \qquad S^*(a, n) = (-1)^n(-a, n)$$

$\mathcal{B}$ is a Hopf algebra isomorphic to $\mathcal{R}$. We call $\mathcal{B}$ the *binomial Hopf algebra* *over the complex numbers*. Note that the vector subspace of $\mathcal{B}$ generated by the elements of the form $(0, k)$, for k in $\mathbb{N}$, is a Hopf algebra isomorphic to $\mathcal{P}$.

5. REALIZATIONS OF $\mathcal{B}$

We present examples of vector spaces that can be equipped with a Hopf algebra structure isomorphic to that of $\mathcal{B}$, and which appear in a natural way in analysis.

Suppose $\mathcal{G}$ is a complex vector space with a basis $G = \{g_{a,k}: a \in \mathbb{C}, k \in \mathbb{N}\}$. The obvious bijection between the bases G of $\mathcal{G}$ and $\mathbb{C} \times \mathbb{N}$ of $\mathcal{B}$ yields a vector space isomorphism between $\mathcal{G}$ and $\mathcal{B}$, and then we can transfer the Hopf algebra structure from $\mathcal{B}$ to $\mathcal{G}$ in the obvious way. We will use the same symbols to denote the operations, the counit, and the antipode on $\mathcal{G}$ and on $\mathcal{B}$.

Let t be a complex variable and define the functions

$$f_{a,k}(t) = \frac{t^k}{k!} e^{at}, \qquad a \in \mathbb{C}, \quad k \in \mathbb{N}$$

We denote by $\mathcal{E}$ the complex vector space generated by the $f_{a,k}$. The elements of $\mathcal{E}$ are called *quasi-polynomials or exponential polynomials* and they are the solutions of linear homogeneous differential equations with constant coefficients. The natural multiplication of the basic elements $f_{a,k}$, considered as functions of t, gives

$$f_{a,k} f_{b,m} = \binom{k+m}{k} f_{a+b,k+m}$$

which coincides with the multiplication $\star$. The coproduct

$$\Gamma f_{a,n} = \sum_{k=0}^{n} f_{a,k} \otimes f_{a,n-k}$$

is induced by the translation maps U_z, defined by $U_z f(t) = f(t + z)$, as in the case of the coproduct Δ on $\mathcal{P}$. The counit on $\mathcal{E}$ and the antipode are

$$\Phi f_{a,n} = f_{a,n}(0) = \delta_{0,n}, \qquad S^* f_{a,n}(t) = f_{a,n}(-t)$$

Therefore $\mathcal{E}$ is a natural Hopf algebra extension of $\mathcal{P}$. Note that the elements $f_{0,k}$, for k in $\mathbb{N}$, form a basis for $\mathcal{P}$.

Consider the linear operators $L_{a,k}$ on $\mathcal{P}$ defined by

$$L_{a,k} = \frac{D^k}{k!} U_a, \qquad a \in \mathbb{C}, \quad k \in \mathbb{N}$$

and let $\mathcal{D}$ be the complex vector space generated by the $L_{a,k}$. We call $\mathcal{D}$ the algebra of shift invariant operators. The multiplication on $\mathcal{D}$ induced by $\mathcal{B}$ coincides with the multiplication of linear operators since $\mathcal{D}$ commutes with translations and $U_a U_b = U_{a+b}$. The coproduct is induced by Leibniz' rule for differentiation,

$$\Gamma L_{a,n} = \sum_{k=0}^{n} L_{a,k} \otimes L_{a,n-k}$$

The counit and the antipode are given by

$$\Phi L_{a,n} = \frac{D^n}{n!} U_a 1 = \delta_{0,n}, \qquad S^* L_{a,n} = (-1)^n \frac{D^n}{n!} U_{-a}$$

Since the translations on $\mathcal{P}$ can be written in the form

$$U_a = \sum_{k \geq 0} \frac{a^k}{k!} D^k = e^{aD}$$

we can consider $\mathcal{D}$ as the Hopf algebra obtained from $\mathcal{E}$ replacing t by the operator D, that is, we can identify $L_{a,k}$ with $f_{a,k}(D)$.

Let $\mathcal{S}$ be the complex vector space generated by the sequences

$$s_{a,k}(n) = \binom{n}{k} a^{n-k}, \qquad (a, k) \in \mathbb{C} \times \mathbb{N}, \quad n \in \mathbb{N}$$

The elements of $\mathcal{S}$ are called *linearly recurrent sequences*. They are the solutions of homogeneous linear difference equations with constant coefficients. The multiplication $\star$ in $\mathcal{S}$ gives

$$s_{a,k}(n) \star s_{b,m}(n) = \binom{k+m}{k} \binom{n}{k+m} (a+b)^{n-k-m}$$

A simple computation using the binomial formula shows that this multiplication coincides with Cauchy's convolution of sequences, defined by

$$f \star g(n) = \sum_{j=0}^{n} \binom{n}{j} f(j) g(n-j)$$

By the Chu–Vandermonde convolution formula, we have

$$\begin{aligned} s_{a,k}(n+m) &= \binom{n+m}{k} a^{n+m-k} \\ &= \sum_{j=0}^{k} \binom{n}{j} a^{n-j} \binom{m}{k-j} a^{m-k+j} \\ &= \sum_{j=0}^{k} s_{a,j}(n) s_{a,k-j}(m) \end{aligned}$$

and hence the coproduct Γ on $\mathcal{S}$ is induced by translations.

The counit and the antipode are

$$\Phi s_{a,k} = \delta_{0,k} = s_{a,k}(0)$$

$$S^* s_{a,k}(n) = (-1)^k \binom{n}{k} (-a)^{n-k} = (-1)^n \binom{n}{k} a^{n-k}$$

Therefore

$$S^* s_{a,k}(n) = (-1)^n s_{a,k}(n)$$

The Hopf algebra $\mathcal{S}$ has been studied by Cerlienco *et al.* (1987), Chin and Goldman (1993), and Peterson and Taft (1980).

Consider now the linear space $\mathcal{L}$ generated by the functions

$$l_{a,k}(t) = \frac{(\log t)^k}{k!} e^{a \log t}$$

where t is a complex variable and we take the principal determination of log t, with imaginary part in the interval $[0, 2\pi)$. Here, as in the case of $\mathcal{E}$, the multiplication $\star$ coincides with the usual multiplication of functions of t. The comultiplication Γ is induced by the map that sends t to tu, since

$$l_{a,k}(tu) = \sum_{j=0}^{k} l_{a,j}(t) l_{a,k-j}(u)$$

The counit Φ is evaluation at $t = 1$,

$$\Phi l_{a,k}(t) = l_{a,k}(1) = \delta_{0,k},$$

the antipode is

$$S^* l_{a,k}(t) = l_{a,k}(1/t)$$

6. CONVOLUTIONS

Let us observe that for some of the concrete realizations of $\mathcal{B}$ described above, the multiplication $\star$ is not the natural multiplication. This happens for $\mathcal{S}$ and for $\mathcal{R}$, where $\star$ is not the multiplication of the objects considered as complex-valued functions in some domain. The natural multiplication in $\mathcal{S}$ is the Hadamard multiplication of sequences, denoted by $\bullet$ and defined by

$$f \bullet g(n) = f(n)g(n), \qquad n \in \mathbb{N}, \quad f, g \in \mathcal{S}$$

The Hadamard product of two basic elements of $\mathcal{S}$ is

$$(s_{a,k} \bullet s_{b,m})(n) = \binom{n}{k} a^{n-k} \binom{n}{m} b^{n-m}$$

We can find an explicit expression for this product as a linear combination of basic elements of $\mathcal{S}$ using the following property of the binomial polynomials.

Proposition 6.1. For any complex number t and natural numbers k and m, with $k \leq m$, we have

$$\binom{t}{k}\binom{t}{m} = \sum_{j=0}^{k} \binom{k+m-j}{j,\, k-j,\, m-j} \binom{t}{k+m-j}$$

This linearization formula is easily proved using induction on k and the basic recurrences for the binomial polynomials. A direct application of Newton's interpolation formula gives another proof.

Corollary 6.1. Let r be the minimum of k and m. Then

$$s_{a,k} \bullet s_{b,m} = \sum_{j=0}^{r} \binom{k+m-j}{j,\, k-j,\, m-j} a^{m-j} b^{k-j} s_{ab,k+m-j}$$

The multiplicaton of elements of $\mathcal{R}$ considered as functions of t is given in (2.20) in the case $a \neq b$, and

$$r_{a,k}(t) r_{a,m}(t) = r_{a,k+m+1}(t)$$

These formulas define a multiplication on any concrete realization $\mathcal{G}$ of $\mathcal{B}$, replacing the $r_{a,j}$ by the corresponding basic elements of $\mathcal{G}$. On the Hopf algebra $\mathcal{E}$, such multiplication coincides with the classic convolution product

$$(f * g)(t) = \int_0^t f(y) g(t-y)\, dy, \qquad t \in \mathbb{R}$$

and which is easily extended to the case of complex values of t.

In view of the above examples, we can obtain many different multiplications, or convolution products, on any vector space isomorphic to $\mathcal{B}$, constructing a vector space $\mathcal{F}$ isomorphic to $\mathcal{B}$ whose elements are complex-valued functions defined on some set. The natural multiplication on $\mathcal{F}$ can be transferred to any other isomorphic copy of $\mathcal{B}$.

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